

Absorbing boundary conditions for Einstein's field equations

Olivier Sarbach

*Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo,
Edificio C-3, Cd. Universitaria. C. P. 58040 Morelia, Michoacán, México*

(Dated: February 1, 2008)

A common approach for the numerical simulation of wave propagation on a spatially unbounded domain is to truncate the domain via an artificial boundary, thus forming a finite computational domain with an outer boundary. Absorbing boundary conditions must then be specified at the boundary such that the resulting initial-boundary value problem is well posed and such that the amount of spurious reflection is minimized. In this article, we review recent results on the construction of absorbing boundary conditions in General Relativity and their application to numerical relativity.

I. INTRODUCTION

To numerically simulate the evolution of hyperbolic partial differential equations on a spatially unbounded domain, one usually replaces the unbounded domain with a finite, compact domain Ω with artificial outer boundary $\partial\Omega$. Boundary conditions on $\partial\Omega$ must then be specified in order to obtain a unique Cauchy evolution. These conditions should be formulated so that they form a well posed initial boundary value problem (IBVP) and, ideally, are completely transparent to the physical problem on the unbounded domain. In practice, complete transparency cannot be achieved easily and therefore, the boundaries introduce spurious reflections into the computational domain Ω . The idea then is to develop what is called absorbing boundary conditions which form a well posed IBVP and insure that the amount of spurious reflection from $\partial\Omega$ is *as small as possible*.

There has been a substantial amount of work on the construction of absorbing (also called non-reflecting in the literature) boundary conditions for wave problems in acoustics, electromagnetism, meteorology, and solid geophysics (see [1] for a review). One approach is based on a hierarchy of *local* boundary conditions [2, 3, 4] with increasing order of accuracy. Although higher order local boundary conditions usually involve solving a high order differential equation at the boundary, the problem can be dealt with by introducing auxiliary variables at the boundary surface [5, 6]. A different approach is based on fast converging series expansions of *exact nonlocal* boundary conditions (see [7] and references therein).

Constructing absorbing outer boundary conditions in General Relativity is difficult. First of all, Einstein's field equations determine the evolution of the metric tensor, so one does not know the geometrical structure of the spacetime before actually solving the IBVP. Hence, it is not clear *a priori* how the geometry of the outer boundary evolves. One could turn the argument around to guide the choice of boundary conditions so that they fix the embedding of the boundary surface into the resulting spacetime. Second, in the Cauchy formulation of Einstein's field equations, one usually encounters constraint fields which propagate *across* the boundary into the computational domain. This is in contrast to the standard Cauchy formulation of Maxwell's equations, where the evolution equations imply that the constraint variables (namely, the divergence of the electric and magnetic fields) are constant in time. As a consequence, in the Einstein case, constraint-preserving boundary conditions need to be specified at $\partial\Omega$ such that constraint violations are not introduced into the computational domain. Finally, due to the nonlinear nature of the theory and its diffeomorphism invariance, it is difficult to define precisely what is meant by in- and outgoing radiation in General Relativity. Therefore, it is not even clear how to quantify the amount of spurious gravitational reflection from the boundary. These issues all contribute to the challenge of developing accurate absorbing boundary conditions for Einstein's field equations.

In this article, we briefly review some recent results on the construction of absorbing outer boundaries in General Relativity. We start with a discussion of absorbing outer boundaries for the wave equation on Minkowski and Schwarzschild spacetimes in Sect. II where we review results from the literature and derive asymptotic outgoing wave solutions on weakly curved spacetimes. Recent attempts to generalize these results to Einstein's field equations are mentioned in Sect. III. In particular, we discuss constraint-preserving boundary conditions, boundary conditions designed to minimize spurious reflections of gravitational waves, recent results on well posed initial-boundary value formulations and recent applications to numerical relativity. Concluding remarks are drawn in Sect. IV.

There exist other approaches for dealing with gravitational wave propagation on an infinite domain which are not discussed in this article. These involve matching techniques (see Ref. [8] for a review) or avoiding introducing an artificial outer boundary altogether by compactifying spatial infinity [9, 10], or making use of hyperboloidal slices and compactifying null infinity (see, for instance, [11, 12, 13]).

II. ABSORBING BOUNDARY CONDITIONS FOR THE WAVE EQUATION

To illustrate some of the ideas involved in constructing absorbing boundary conditions, we start with three examples in order of increasing difficulty. The simplest example is the flat wave equation on an interval $(-1, 1)$. In this case, perfectly absorbing boundary conditions can be constructed since the most general solution is simply the superposition of an arbitrary function of retarded time with an arbitrary function of advanced time. The second example discusses the flat wave equation on a three-dimensional ball B_R of radius $R > 0$. This example is already much more difficult than the previous one since waves can now travel in infinitely many directions. One strategy here is to obtain a hierarchy of absorbing boundary conditions with increasing order of accuracy. The last example generalizes the second example to a weakly curved background. When the curvature of the background is taken into account, a wave solution propagating on Minkowski spacetime acquires two types of correction terms. The first is a curvature correction term which obeys Huygens' principle and the second is a fast decaying term which violates Huygens' principle and describes the backscatter off curvature. It is important to understand these effects before constructing absorbing boundaries. An accurate boundary condition must not eliminate the backscatter which is a real, physical effect! This example shows that an absorbing boundary condition (as defined in the introduction) does not mean that *all* reflections off the boundary are to be eliminated, but only *spurious* reflections.

A. The flat wave equation on an interval

Consider the one-dimensional flat wave equation

$$(\partial_t^2 - \partial_x^2) u(t, x) = 0, \quad t > 0, x \in [-1, 1]. \quad (1)$$

The general solution is a superposition of a left- and a right-moving solution,

$$u(t, x) = f_{\searrow}(x + t) + f_{\nearrow}(x - t).$$

Therefore, the boundary conditions

$$(\partial_t - \partial_x)u(t, -1) = 0, \quad (\partial_t + \partial_x)u(t, +1) = 0, \quad t > 0, \quad (2)$$

are perfectly absorbing according to our terminology. Indeed, the operator $b_1 := \partial_t + \partial_x$ has as its kernel the right-moving solutions $f_{\nearrow}(x - t)$, hence the boundary condition $b_1 u(t, 1) = 0$, $t > 0$, does not “touch” these solutions. On the other hand, $b_1 f_{\searrow}(t + x) = 2f'_{\searrow}(t + x)$, so the boundary condition at $x = 1$ requires that $f_{\searrow}(v) = f_{\searrow}(1)$ is constant for advanced time $v = t + x > 1$. A similar argument shows that the left boundary condition implies that $f_{\nearrow}(-u) = f_{\nearrow}(-1)$ is constant for retarded time $u = t - x > 1$. Furthermore, it is known that the conditions (2) together with Eq. (1) and suitable initial conditions for u and $\partial_t u$ at $t = 0$ yield a well posed IBVP. In particular, the solution is identically zero after one crossing time ($t \geq 2$) if the initial data has compact support.

B. The flat wave equation on a three-dimensional ball

Next, consider the three-dimensional flat wave equation

$$(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2) u(t, x) = 0, \quad t > 0, (x, y, z) \in B_R, \quad (3)$$

on a ball B_R of radius $R > 0$. The general solution can be decomposed into spherical harmonics

$$u(t, r, \vartheta, \varphi) = \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell m}(t, r) Y^{\ell m}(\vartheta, \varphi)$$

which yields the family of reduced equations

$$\left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2} \right] u_{\ell m}(t, r) = 0, \quad t > 0, r \in (0, R). \quad (4)$$

For $\ell = 0$ this equation reduces to the previous example, and the general solution is $u_{00}(t, r) = U_{00\searrow}(r-t) + U_{00\searrow}(r+t)$ with $U_{00\searrow}$ and $U_{00\searrow}$ two arbitrary functions. Therefore, the boundary condition

$$\mathcal{B}_0 : \quad b(ru)|_{r=R} = 0, \quad (5)$$

where $b := r^2(\partial_t + \partial_r)$ is perfectly absorbing for spherical waves. For $\ell \geq 1$, exact solutions can be generated from the solutions for $\ell = 0$ by applying suitable differential operators to $u_{00}(t, r)$. For this, we define the operators [14]

$$a_\ell \equiv \partial_r + \frac{\ell}{r}, \quad a_\ell^\dagger \equiv -\partial_r + \frac{\ell}{r}$$

which satisfy the operator identities

$$a_{\ell+1}a_{\ell+1}^\dagger = a_\ell^\dagger a_\ell = -\partial_r^2 + \frac{\ell(\ell+1)}{r^2}.$$

As a consequence, for each $\ell = 1, 2, 3, \dots$, we have

$$\begin{aligned} \left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2} \right] a_\ell^\dagger a_{\ell-1}^\dagger \dots a_1^\dagger &= \left[\partial_t^2 + a_\ell^\dagger a_\ell \right] a_\ell^\dagger a_{\ell-1}^\dagger \dots a_1^\dagger \\ &= a_\ell^\dagger \left[\partial_t^2 + a_{\ell-1}^\dagger a_{\ell-1} \right] a_{\ell-1}^\dagger \dots a_1^\dagger \\ &= a_\ell^\dagger a_{\ell-1}^\dagger \dots a_1^\dagger \left[\partial_t^2 - \partial_r^2 \right]. \end{aligned}$$

Therefore, we have the explicit in- and outgoing solutions

$$\begin{aligned} u_{\ell m \searrow}(t, r) &= a_\ell^\dagger a_{\ell-1}^\dagger \dots a_1^\dagger V_{\ell m}(r+t) = \sum_{j=0}^{\ell} (-1)^j \frac{(2\ell-j)!}{(\ell-j)! j!} (2r)^{j-\ell} V_{\ell m}^{(j)}(r+t), \\ u_{\ell m \nearrow}(t, r) &= a_\ell^\dagger a_{\ell-1}^\dagger \dots a_1^\dagger U_{\ell m}(r-t) = \sum_{j=0}^{\ell} (-1)^j \frac{(2\ell-j)!}{(\ell-j)! j!} (2r)^{j-\ell} U_{\ell m}^{(j)}(r-t), \end{aligned} \quad (6)$$

where $V_{\ell m}$ and $U_{\ell m}$ are arbitrary smooth functions with j 'th derivatives $V_{\ell m}^{(j)}$ and $U_{\ell m}^{(j)}$, respectively. In order to construct boundary conditions which are perfectly absorbing for $u_{\ell m}$, one first shows the following identity. Let $b = r^2(\partial_t + \partial_r)$ as above, then

$$b^{\ell+1} a_\ell^\dagger a_{\ell-1}^\dagger \dots a_1^\dagger U(r-t) = 0 \quad (7)$$

for all $\ell = 0, 1, 2, \dots$ and all sufficiently smooth functions U . This identity follows easily from Eq. (6) and the fact that $b^{\ell+1}(r^k) = k(k+1) \dots (k+\ell)r^{k+\ell+1} = 0$ if $k \in \{0, -1, -2, \dots, -\ell\}$. Therefore, given $L \in \{1, 2, 3, \dots\}$, the boundary condition

$$\mathcal{B}_L : \quad b^{L+1}(ru)|_{r=R} = 0, \quad (8)$$

leaves the outgoing solutions with $\ell \leq L$ unaltered. Notice that this condition is *local* in the sense that its formulation does not require the decomposition of u into spherical harmonics. Furthermore, it was shown in [3] for domains which can be more general than B_R that each condition \mathcal{B}_L yields a well posed IBVP. By uniqueness this implies that initial data corresponding to a purely outgoing solution with $\ell \leq L$ yields a purely outgoing solution (without reflections). In this sense, the condition \mathcal{B}_L is *perfectly absorbing for waves with $\ell \leq L$* . For waves with $\ell > L$, one obtains spurious reflections; however, for monochromatic radiation with wave number k , the corresponding amplitude reflection coefficients can be calculated to decay as $(kR)^{-2(L+1)}$ in the wave zone $kR \gg 1$ [15]. Furthermore, in most scenarios with smooth solutions, the amplitudes corresponding to the lower few ℓ 's will dominate over the ones with high ℓ so that reflections from high ℓ 's are unimportant. For a numerical implementation of the boundary condition \mathcal{B}_2 via spectral methods and a possible application to General Relativity see Ref. [16].

C. The wave equation on a weakly curved background

Next, we generalize the previous example to the wave equation on a weakly curved background. More specifically, we consider the wave equation on the far field region of an asymptotically flat spacetime. Such a spacetime has the form of Minkowski plus $1/r$ correction terms, where r is the areal radius. The monopolar correction is given by the M/r term of the Schwarzschild metric, where M represents the total (Arnowitt-Deser-Misner) mass of the spacetime. Therefore, to first approximation, we may assume that spacetime is described by the exterior of a Schwarzschild metric

of mass M . In outgoing Eddington-Finkelstein coordinates $(t - r, r)$ the reduced wave equation on a Schwarzschild background is

$$\left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2} \right] u_{\ell m} = -\frac{2M}{r} \left[(\partial_t + \partial_r)^2 - \frac{1}{r}(\partial_t + \partial_r) + \frac{1+\sigma}{r^2} \right] u_{\ell m}. \quad (9)$$

Here, σ is a parameter which in the present case is zero but is left arbitrary for future convenience. For $M = 0$, this equation reduces to the flat reduced wave equation (4), for which outgoing solutions have the form

$$u_{\ell m \nearrow}^{(0)}(t, r) = a_\ell^\dagger a_{\ell-1}^\dagger \dots a_1^\dagger U_{\ell m}(r - t)$$

with a sufficiently smooth function $U_{\ell m}$. For the following, we also assume that $U_{\ell m}$ is bounded and vanishes for sufficiently negative values of its argument. If $M > 0$, we seek an approximate outgoing solution in the far field region $r \approx R \gg M$ of the form

$$u_{\ell m \nearrow}(t, r) = u_{\ell m \nearrow}^{(0)}(t, r) + \frac{2M}{R} u_{\ell m \nearrow}^{(1)}(t, r) + O\left(\frac{2M}{R}\right)^2. \quad (10)$$

Plugging this into Eq. (9) and expanding in M/R , we find that $u_{\ell m \nearrow}^{(1)}$ must satisfy

$$\begin{aligned} \left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2} \right] u_{\ell m \nearrow}^{(1)} &= -\frac{R}{r} \left[(\partial_t + \partial_r)^2 - \frac{1}{r}(\partial_t + \partial_r) + \frac{1+\sigma}{r^2} \right] u_{\ell m \nearrow}^{(0)} \\ &= -\frac{R}{r^3} \sum_{j=0}^{\ell} (-1)^j \frac{(2\ell-j)!}{(\ell-j)!j!} [(\ell+1-j)^2 + \sigma] (2r)^{j-\ell} U_{\ell m}^{(j)}(r-t). \end{aligned} \quad (11)$$

The solution to this equation can be written as the sum of two terms,

$$u_{\ell m \nearrow}^{(1)}(t, r) = u_{\ell m}^{(curv)}(t, r) + u_{\ell m}^{(backscatter)}(t, r). \quad (12)$$

The first term includes corrections from the curvature and obeys Huygens' principle. It has the form

$$u_{\ell m}^{(curv)}(t, r) = R \sum_{j=0}^{\ell} (-1)^j \frac{(2\ell-j)!}{(\ell-j)!j!} (2r)^{j-\ell} c_j U_{\ell m}^{(j+1)}(r-t),$$

where the coefficients c_0, c_1, \dots, c_ℓ can be computed from the recursion relation

$$\begin{aligned} c_\ell &= 0, \quad c_{\ell-1} = 0, \quad c_{\ell-2} = -\frac{2(1+\sigma)}{(\ell-1)\ell(\ell+1)(\ell+2)}, \\ c_{j-1} &= c_j - \frac{2(\ell-j)}{2\ell-j} \frac{(\ell-j)^2 + \sigma}{j(j+1)(2\ell+1-j)}, \quad j = \ell-2, \ell-3, \dots, 1. \end{aligned} \quad (13)$$

The second term is a fast decaying term which violates Huygens' principle and describes the backscatter off the curvature of the background. It has the form

$$u_{\ell m}^{(backscatter)}(t, r) = R(1 - \delta_\ell) \frac{(2\ell)!}{(\ell+1)!} \frac{\ell^2 + \sigma}{(2r)^{\ell+1}} U_{\ell m}(r-t) + R(1 + \delta_\ell \sigma) \int_{r-t}^{\infty} K_\ell(t, r, x) U_{\ell m}(x) dx, \quad (14)$$

where $\delta_\ell = 1$ for $\ell = 0$ and $\delta_\ell = 0$ for $\ell \geq 1$, and the integral kernel $K_\ell(t, r, x)$ is given by

$$K_\ell(t, r, x) = a_\ell^\dagger a_{\ell-1}^\dagger \dots a_1^\dagger \frac{1}{(t+r+x)^2} = \frac{1}{(2r)^{2+\ell}} \sum_{j=0}^{\ell} \frac{(2\ell-j)!}{(\ell-j)!} (j+1) z^{-2-j} \Big|_{z=\frac{t+r+x}{2r}}.$$

It is not difficult to verify that the expression given in Eq. (12) indeed solves Eq. (11) if one notes the recursion relation (13) and the following properties

$$\left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2} \right] K_\ell(t, r, x) = 0, \quad r^2(\partial_t + \partial_r) K_\ell(t, r, r-t) = -\frac{(2\ell+1)!}{\ell!} \frac{1}{(2r)^{\ell+1}},$$

of the integral kernel. Of course, the expression given in Eq. (12) is not the unique solution to Eq. (11); one can add an arbitrary homogeneous solution to it. However, the solution is uniquely characterized by the following conditions:

$$\begin{aligned} \lim_{r \rightarrow \infty} u_{\ell m \nearrow}^{(1)}(\text{const.} + r, r) &= 0 & (u_{\ell m \nearrow}^{(1)} \text{ vanishes at future null infinity}), \\ \lim_{r \rightarrow \infty} u_{\ell m \nearrow}^{(1)}(\text{const.} - r, r) &= 0 & (u_{\ell m \nearrow}^{(1)} \text{ vanishes at past null infinity}), \\ \lim_{t \rightarrow \infty} u_{\ell m \nearrow}^{(1)}(t, \text{const.}) &= 0 & (u_{\ell m \nearrow}^{(1)} \text{ vanishes at future time-like infinity}). \end{aligned}$$

Notice that for $\ell \geq 1$, the third requirement is necessary in order to exclude homogeneous solutions of the form $u_{\ell m \nearrow}^{(1)}(t, r) = (c_1 t + c_0) r^{-\ell}$ (where c_0 and c_1 are some constants), which vanish at both future and past null infinity.

Summarizing, outgoing wave solutions have the form

$$u_{\ell m \nearrow}(t, r) = \sum_{j=0}^{\ell} (-1)^j \frac{(2\ell - j)!}{(\ell - j)! j!} (2r)^{j-\ell} \left[U_{\ell m}^{(j)}(r - t) + 2M c_j U_{\ell m}^{(j+1)}(r - t) \right] + \frac{2M}{R} u_{\ell m}^{(\text{backscatter})}(t, r) + O\left(\frac{2M}{R}\right)^2, \quad (15)$$

where the coefficients c_j are given in Eq. (13) and $u_{\ell m}^{(\text{backscatter})}$ is given in Eq. (14). A systematic derivation which includes the correction terms in M/R of arbitrarily high order is given in [17].

Now let us analyze how much spurious reflection is introduced from these outgoing wave solutions when the condition \mathcal{B}_L is imposed at the boundary surface. If for the moment we neglect correction terms arising from the backscatter as well as terms which are quadratic or higher order in M/R , then by the same arguments as in the previous subsection we conclude that \mathcal{B}_L is perfectly absorbing for outgoing waves with angular momentum number ℓ smaller or equal than L . Therefore, \mathcal{B}_L automatically takes care of the curvature correction terms $u_{\ell m}^{(\text{curv})}$. If effects from backscatter are taken into account, \mathcal{B}_L is not perfectly absorbing for waves with $\ell \leq L$ anymore, however, in this case spurious reflections off the boundary surface are very small. In order to quantify this statement, consider for instance monopolar scalar radiation ($\ell = 0$, $\sigma = 0$) for which

$$u_{00 \nearrow}(t, r) = U_{00}(r - t) + 2M \int_{r-t}^{\infty} \frac{U_{00}(x) dx}{(t + r + x)^2} + O\left(\frac{2M}{R}\right)^2. \quad (16)$$

Since

$$(\partial_t + \partial_r) u_{00 \nearrow}(t, r) = -8M \int_{r-t}^{\infty} \frac{U_{00}(x) dx}{(t + r + x)^3} + O\left(\frac{2M}{R}\right)^2 = -\frac{2M}{r^2} \int_0^{\infty} \frac{U_{00}(r - t + 2ry) dy}{(1 + y)^3} + O\left(\frac{2M}{R}\right)^2,$$

the boundary condition \mathcal{B}_0 is not perfectly absorbing unless $M = 0$. As a consequence, the solution of Eq. (9) consists of a superposition of an in- and an outgoing wave:

$$u_{00}(t, r) = u_{00 \nearrow}(t, r) + u_{00 \searrow}(t, r),$$

where the ingoing wave has the form $u_{00 \searrow}(t, r) = V_{00}(r + t) + O(2M/R)$ with V_{00} a smooth function. Consider monochromatic waves of the form

$$U_{00}(r - t) = e^{ik(t-r)}, \quad V_{00}(r + t) = \gamma e^{-ik(r+t)}, \quad (17)$$

where k is a given wave number and γ is an amplitude reflection coefficient. For the following, we assume that $0 < k \ll M^{-1}$. If kM is of the order of unity or larger, powers of kR which are multiplied by $(2M/R)^2$ might be comparable in size or larger than terms of the form M/R times unity, and in this case the $(2M/R)^2$ -correction terms in $u_{00 \nearrow}(t, r)$ might actually be larger than the $(2M/R)$ -correction term. Imposing the boundary condition \mathcal{B}_0 at $r = R$, the ansatz (17) yields

$$\gamma = -\frac{M}{R} \frac{e^{2ikR}}{ikR} \int_0^{\infty} \frac{e^{2ikRy} dy}{(1 + y)^3} + O\left(\frac{2M}{R}\right)^2.$$

It can be shown that the integral decays as $(kR)^{-1}$ for large kR . Therefore, under the assumption that $M \ll k^{-1} \ll R$ we find that $|\gamma|$ decays as $(M/R)(kR)^{-2}$. Using similar arguments it can be shown that the boundary condition \mathcal{B}_L

yields a reflection coefficient that decays as $(M/R)(kR)^{-(L+2)}$ or faster for monochromatic waves satisfying $\ell \leq L$ and $M \ll k^{-1} \ll R$.

Finally, we remark that it is, in principle, possible to improve the boundary conditions \mathcal{B}_L in order to take into account the first order correction terms in M/R of the backscatter. However, by the very nature of the backscatter, such boundary conditions cannot be local anymore. As an example, consider again monopolar scalar radiation where outgoing solutions have the form (16). Then, the boundary condition [18]

$$(\partial_t + \partial_r)u_{00}(t, R) + \frac{2M}{R^2} \int_0^{t/2R} \frac{u_{00}(t - 2Ry, R)dy}{(1+y)^3} = G(t), \quad t \geq 0 \quad (18)$$

with the boundary data

$$G(t) = -\frac{2M}{R^2} \int_{t/2R}^{\infty} \frac{u_{00}(0, R - t + 2Ry)dy}{(1+y)^3},$$

is perfectly absorbing up to (and including) order $2M/R$. Notice that the integral on the left-hand side of Eq. (18) only involves the past portion $\{(\tau, R) : 0 \leq \tau \leq t\}$ of the boundary which is available from the past history of a Cauchy evolution starting at $t = 0$. The boundary data G involves an integral over the initial data at $t = 0$ over the region $r > R$ exterior to the computational domain and takes care of the backscatter that occurred in the past $t < 0$. If the initial data is compactly supported in the interval $(0, R)$ this integral is zero and can be discarded. It was shown in [18] that the boundary condition (18) is stable in the sense that it admits an energy estimate. This construction can be repeated for waves with arbitrary angular momentum number ℓ .

A different method for constructing absorbing boundary conditions for linearized gravitational waves propagating on a Schwarzschild background has recently been presented in [19, 20, 21]. This method is based on fast converging series expansions of an *exact nonlocal* boundary condition and takes into account arbitrarily high correction terms in M/R of the Schwarzschild metric. However, there is no advantage to obtaining a boundary condition which takes into account the exact form of the Schwarzschild metric beyond the order of M/R in the construction of boundary conditions for wave propagation on an asymptotically flat curved background. The reason for this is that a generic, asymptotically flat background only agrees with the Schwarzschild metric up to order M/R . If second order effects are to be taken into account, quadratic terms in M/R and linear terms in J/R^2 (where J is the total angular momentum of the background) from the background metric must be considered.

III. ABSORBING BOUNDARY CONDITIONS FOR EINSTEIN'S FIELD EQUATIONS

The construction of absorbing outer boundary conditions in General Relativity is much more difficult than for the wave equation on a *fixed background* discussed in the previous section. At least three additional complications arise. First, in the Cauchy problem of General Relativity, Einstein's field equations split into a set of evolution equations and a set of constraints. If the spatial time slices are infinite or compact without boundaries it can be shown via the use of Bianchi's identities that any smooth enough solution of the evolution equations with constraint satisfying initial data automatically satisfies the constraints everywhere and at all times. However, if the time slices possess a nonempty boundary, this statement holds only if constraint-preserving boundary conditions are specified. The second complication is due to the fact that gravitational waves do not propagate on a fixed background but deform the spacetime metric as they evolve. As a consequence, it is not clear how to "fix" the boundary geometrically. It would be nice if one could specify the boundary conditions in such a way that the embedding of the boundary surface in the resulting spacetime is independent of the coordinate choice for which the evolution is performed. Otherwise, two evolutions using different coordinates might obtain different portions of spacetime even if they both start with the same initial slice and data. Finally, the third complication stems from the nonlinear nature of Einstein's field equations. In particular, the superposition principle for wave propagation does not hold so it is much harder to superpose an outgoing and an ingoing wave as was done in the previous section in order to quantify the amount of spurious reflection.

A. Constraint-preserving boundary conditions

The construction of constraint-preserving boundary conditions is probably the best understood and most studied issue of all the complications listed above: see Refs. [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37,

38, 39, 40, 41, 42, 43, 44] for analytic studies and Refs. [24, 25, 28, 36, 37, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54] for numerical studies. The basic idea in constructing constraint-preserving boundary conditions is to derive the constraint propagation system, describing the propagation of constraint violations, and to impose boundary conditions for this system which ensure that zero is the only solution with trivial initial data. As an example, consider Einstein’s field equations in harmonic coordinates,

$$\Gamma^a := \square_g x^a = 0,$$

where \square_g denotes the d’Alembertian operator with respect to the metric g_{ab} . In these coordinates, Einstein’s vacuum equations reduce to a set of ten coupled, quasilinear wave equations of the form

$$g^{cd}\partial_c\partial_d g_{ab} = F_{ab}(g)[\partial g, \partial g],$$

subject to the constraint $\Gamma^a = 0$, where $F_{ab}(g)[\partial g, \partial g]$ depends quadratically on the first derivatives of the metric fields g_{ab} . Using the twice contracted Bianchi identities, one finds that as a consequence of the evolution equations, Γ^c satisfies a wave equation on its own,

$$g^{cd}\partial_c\partial_d\Gamma^a = L^a(g, \partial g, \partial^2 g)[\Gamma]$$

where $L^a(g, \partial g, \partial^2 g)[\Gamma]$ is a *linear first order differential operator* in Γ^a with coefficients depending on g_{ab} and its partial derivatives up to second order. Therefore, the initial conditions $\Gamma^a|_{t=0} = 0$, $\partial_t\Gamma^a|_{t=0} = 0$ insure that $\Gamma^a = 0$ on the domain of dependence of the initial slice. If the initial slice is a Cauchy slice, this implies that $\Gamma^a \equiv 0$ everywhere on the spacetime, but if time-like boundaries are present, one needs to impose additional conditions at the boundary in order to guarantee $\Gamma^a = 0$ everywhere. There are different ways of assuring that $\Gamma^a \equiv 0$ is the only solution with trivial initial data $\Gamma^a|_{t=0} = 0$, $\partial_t\Gamma^a|_{t=0} = 0$. The simplest way is to specify Dirichlet data [41]

$$\Gamma^a|_{\partial\Omega} = 0.$$

However, other conditions, such as Sommerfeld-type conditions [39, 42, 54, 55] or higher-order absorbing boundary conditions [56] are also possible.

B. Gauge-controlling boundary conditions

In the above example of Einstein’s equations in harmonic coordinates, where the evolution equations have the form of ten coupled wave equations, one needs ten boundary conditions. As illustrated above, four conditions are needed in order to insure constraint propagation. Since there are two gravitational degrees of freedom, one expects that two boundary conditions are needed in order to control gravitational radiation. The remaining four boundary conditions are related to the residual freedom in choosing harmonic coordinates and fixing the geometry of the boundary surface. Preliminary ideas about how to specify such “gauge” controlling boundary conditions are given in [54, 56], but it is not clear yet how these conditions can be used in order to fix (at least part of) the geometry of the boundary surface. An exception are the boundary conditions constructed in [23] for a tetrad formulation of Einstein’s vacuum equations, which specify the mean curvature of the boundary surface as embedded in spacetime to be an arbitrary constant.

Perhaps even more important than fixing the geometry of the outer boundary is the ability to specify a unique radial coordinate r and a unique outward radial null vector l^a at each point of the boundary. Such quantities are needed for the generalization of the hierarchy of boundary conditions \mathcal{B}_L defined in Eq. (8), where $b = r^2 l^a \nabla_a$. A recent proposal for constructing r and l^a based on the assumption that near the boundary, spacetime can be represented as Schwarzschild plus a small perturbation thereof is given in [18]. However, it is not yet completely clear how to identify the Schwarzschild background in this proposal.

C. Absorbing wave boundary conditions

Once constraint- and gauge-controlling boundary conditions have been specified, the next step is to construct boundary conditions which control the physical degrees of freedom by minimizing the amount of spurious gravitational reflection off the boundary surface. If the boundary is placed far from the strong field region, the field equations can be linearized about a weakly curved spacetime near the outer boundary. As discussed in Sect. II C, it is sufficient to consider a Schwarzschild background provided $R \gg M$ where R is the radius of the outer boundary and M the total (Arnowitt-Deser-Misner) mass of the system. Therefore, near the outer boundary, it is safe to assume that spacetime can be written as Schwarzschild plus a small perturbation thereof.

Linear perturbations on a Schwarzschild background can be described by the Regge-Wheeler-Zerilli formalism [57, 58, 59]. By performing a decomposition of the metric perturbations into spherical tensor harmonics, one obtains in this formalism two families of master equations for gauge-invariant potentials $\Phi_{\ell m}^{(\pm)}$ describing even (+) and odd (−) parity metric fluctuations with angular momentum numbers $\ell \geq 2$ and $|m| \leq \ell$. In particular, the metric perturbations in the Regge-Wheeler gauge can be reconstructed from the potentials $\Phi_{\ell m}^{(\pm)}$ *without solving additional differential equations* [59]. To first order in M/R , the master equations for $\Phi_{\ell m}^{(\pm)}$ have the form (9), where $\sigma^{(+)} = -4 + 3/[(\ell - 1)(\ell + 2)]$ in the even-parity case and $\sigma^{(-)} = -4$ in the odd-parity case. Therefore, approximate outgoing solutions have the form (15) and in principle, the boundary conditions \mathcal{B}_L : $b^{L+1}\Phi_{\ell m}^{(\pm)}|_{r=R} = 0$, where $b = r^2(\partial_t + \partial_r)$, can be applied to the gauge-invariant quantities $\Phi_{\ell m}^{(\pm)}$. However, the relation between $\Phi_{\ell m}^{(\pm)}$ and the metric perturbations is nonlocal in the sense that it depends on the angular momentum number ℓ . Therefore, applying \mathcal{B}_L in this way results in a nonlocal boundary conditions and its implementation requires a decomposition into spherical harmonics at the boundary. An alternative way is to first compute the (linearized) Weyl scalar Ψ_0 from $\Phi_{\ell m}^{(\pm)}$ which is also a gauge-invariant quantity and to formulate the boundary condition on Ψ_0 . If $(t - r, r)$ denote outgoing Eddington-Finkelstein coordinates for the Schwarzschild background, and $\Psi_0 = C_{abcd}l^a m^b l^c m^d$ is constructed from the Weyl tensor C_{abcd} and the null vectors $l^a \partial_a = \partial_t + \partial_r$, $m^a \partial_a = (\sqrt{2}r)^{-1}(\partial_\vartheta + i \sin^{-1} \vartheta \partial_\varphi)$ the boundary conditions [18, 60]¹

$$\mathcal{C}_L : \quad \partial_t [r^2 (\partial_t + \partial_r)]^{L-1} (r^5 \Psi_0) \Big|_{r=R} = 0, \quad L = 1, 2, 3, \dots \quad (19)$$

are perfectly absorbing for linearized gravitational waves with angular momentum number $\ell \leq L$ to first order in M/R if backscatter is neglected. Here, the time derivative operator ∂_t in front of the operator inside the square brackets is introduced in order to allow for a static contribution to Ψ_0 . Notice that for $L = 1$ this condition just freezes Ψ_0 to its initial value. This freezing- Ψ_0 boundary condition has been given before in formulations of the IBVP of Einstein's equations [23, 36, 40, 42, 47]. (Actually, the formulations in Refs. [23, 36, 40] also consider more general boundary conditions which allow to couple Ψ_4 to Ψ_0 but it is unclear if this coupling is useful for reducing spurious reflections.) In this sense, the hierarchy \mathcal{C}_L of boundary conditions improves the freezing- Ψ_0 one. Reflection coefficients due to spurious reflections from backscatter for quadrupolar waves are computed in [18]. It is found that for a spherical outer boundary and quadrupolar gravitational radiation, \mathcal{C}_2 reduces spurious reflections by a factor of $(15M/2R)(kR)^{-2}$ compared to the freezing- Ψ_0 condition \mathcal{C}_1 when $kR > 1$. In [18], a new boundary condition \mathcal{D}_2 similar to the condition (18) is derived which takes into account first order correction terms in M/R of the backscatter for quadrupolar linear waves. It would be interesting to generalize this analysis to take into account second order effects. However, in this case, it is not sufficient to consider perturbations of a Schwarzschild background since quadratic effects in M/R and linear effects in J/R^2 (where J is the total angular momentum of the spacetime) from the full metric should also be included.

D. Well posedness results

Once constraint-preserving absorbing boundary conditions with the desired properties have been specified, the next step is to prove the well posedness of the resulting IBVP. That is, one has to prove that for given initial data u_0 in an appropriate function space there exists a unique solution $u(t)$ of the evolution equations in a time interval $[0, T]$ which satisfies the constructed boundary conditions and such that $u(0) = u_0$. Furthermore, one needs to show that $u(t)$ depends continuously on the initial and boundary data in the sense that if $u^{(n)} \rightarrow u_0$ is a sequence of initial data converging to u_0 then $u^{(n)}(t)$ converges to $u(t)$ for each $t \in [0, T]$. This property is important for the convergence of a numerical approximation since in this case the initial data always contains errors. Finally, one has to check that if u_0 satisfies the constraint equation, so does $u(t)$ for each $t \in [0, T]$.

A well posed IBVP for Einstein's vacuum equations was presented in Ref. [23]. This work, which is based on a tetrad formulation, recasts the evolution equations into first order symmetric hyperbolic quasilinear form with maximally dissipative boundary conditions [61, 62] for which (local in time) well posedness is guaranteed [63]. There has been considerable effort to obtain well posed formulations for the more commonly used metric formulations of gravity. Partial results using similar mathematical techniques as in [23] were obtained in [24, 25, 26, 33, 34, 38, 43, 44].

¹ The normalization of the null vector l^a differs from the one chosen in [18] by a factor of $\sqrt{2/N}$, where $N = 1 - 2M/r$. This explains the absence of the factor N^{-1} in Eq. (19).

However, most of these works are either restricted to the linearized equations or to reflecting (and not absorbing) boundary conditions. For results on coupled hyperbolic-elliptic linear problems with constraint-preserving boundary conditions based on semigroup techniques see [35, 40].

A different technique for showing the well posedness of the IBVP is based on the frozen coefficient principle where one freezes the coefficients of the evolution and boundary operators. In this way, the problem is simplified to a linear, constant coefficient problem on the half-space which can be solved explicitly by using a Fourier-Laplace transformation [64]. This method yields a simple algebraic condition (the determinant condition) which is necessary for the well posedness of the IBVP. Work based on verifying the determinant condition for the Einstein case is given in [22, 27, 33, 34, 36, 42]. Sufficient conditions for the well posedness of the frozen coefficient problem were developed by Kreiss [65]. Kreiss' theorem provides a stronger form of the determinant condition whose satisfaction leads to well posedness if the evolution system is strictly hyperbolic. One of the key results in [65] is the construction of a smooth symmetrizer for the problem for which well posedness can be shown via an energy estimate in the frequency domain. Using the theory of pseudo-differential operators it is expected that the verification of Kreiss' condition also leads to well posedness for quasilinear problems, like Einstein's field equations. Work based on the verification of Kreiss' condition in the Einstein case is given in [22, 42] but since in that case the evolution system is not strictly hyperbolic it is not clear if these results imply well posedness. Recently, Kreiss and Winicour [41] introduced a new pseudo-differential first order reduction of the wave equation which leads to a strictly hyperbolic system. Using this reduction they were able to prove well posedness of the IBVP for Einstein's field equations in harmonic coordinates in the frozen coefficient approximation. This work is generalized to higher-order absorbing boundary conditions in [56]. For an alternative proof of the results in [41] which does not require a pseudo-differential first order reduction see [66].

E. Applications to numerical relativity

For applications of constraint-preserving boundary conditions to numerical relativity, see Refs. [45, 46] for simulations of self-gravitating scalar fields in spherical symmetry, Ref. [47] for the simulation of 1D colliding gravitational plane waves, Ref. [51] for evolutions of Brill waves in axisymmetry, Refs. [24, 25, 28, 36, 37, 39, 50, 52, 53, 54] for tests in three spatial dimensions, Ref. [55] for binary black hole simulations and Refs. [67, 68] for the simulation of bubble spacetimes in five-dimensional theories of gravity. In particular, Refs. [36, 37] implement a first order symmetric hyperbolic formulation of Einstein's vacuum equations with the freezing- Ψ_0 boundary condition, and Refs. [39, 54, 55] also freeze Ψ_0 at the boundary but use the harmonic formulation of the field equations. In [54], constraint-preserving freezing- Ψ_0 boundary conditions are tested for the case of a perturbed Schwarzschild black hole and compared to other types of boundary conditions proposed in the literature. It is found that the version of constraint-preserving freezing- Ψ_0 boundary conditions in [54] performs better than all alternate boundary treatments tested. It should be interesting to numerically implement the boundary conditions \mathcal{C}_L , $L \geq 2$, given in Eq. (19) which are refinements of the freezing- Ψ_0 boundary condition.

IV. CONCLUSIONS

Formulating absorbing outer boundary conditions for the numerical solution of Einstein's field equations involves five steps: i) The construction of constraint-preserving boundary conditions which make sure that no constraint-violating modes enter the computational domain, ii) finding boundary conditions that geometrically control the evolution of the boundary surface, iii) finding conditions that minimize the amount of spurious reflection of gravitational radiation off the boundary, iv) proving well posedness of the resulting initial-boundary value problem (IBVP) and v) discretizing the problem.

As discussed in this article, there has been a lot of effort in carrying out step i) which is, by now, well-understood. In contrast to this, step ii) needs further work. Regarding step iii), a promising approach for minimizing spurious reflections is the hierarchy \mathcal{C}_L of local boundary conditions on the Weyl scalar Ψ_0 presented in [18]. They have the property of being perfectly absorbing including curvature corrections (but neglecting backscatter) to order M/R for all multipoles of gravitational radiation up to L , where M is the Arnowitt-Deser-Misner mass of the spacetime and R a typical radius of the boundary surface. However, their precise formulation requires a radial coordinate and an outward radial null vector field at the boundary whose unambiguous definition is an open problem and could benefit from progress in step ii). Regarding step iv), a complete proof of the well posedness of the IBVP has been given in [23] for a tetrad formulation of Einstein's field equations. Recently, proofs for well posedness have also been given for the frozen coefficient limit of Einstein's equations in harmonic coordinates [41, 56] and it is expected that these results can be generalized to the full Einstein equations based on the theory of pseudo-differential operators. Finally, in step v),

promising results have been achieved in the numerical implementation of the harmonic formulation with constraint-preserving absorbing boundary conditions [39, 53, 55]. It is expected that the boundary condition \mathcal{C}_2 proposed in [18], which is perfectly absorbing for quadrupolar linearized gravitational radiation, will improve these results. Finally, it would be interesting to develop similar boundary conditions for the Baumgarte-Shapiro-Shibata-Nakamura [69, 70] formulation of Einstein's field equations which is often used in numerical relativity. For partial results along these lines, see [32, 34, 71].

Acknowledgments

It is a pleasure to thank J. Bardeen, L. Buchman, L. Lehner, G. Nagy, O. Reula, O. Rinne, M. Tiglio and J. Winicour for many enlightening discussions during my work on boundary conditions. The author also thanks L. Buchman for reading the manuscript and helpful suggestions. This work was partially supported by grant CIC-4.20 to Universidad Michoacana.

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- [1] D. Givoli. Non-reflecting boundary conditions. *J. Comp. Phys.*, 94:1–29, 1991.
 - [2] B. Engquist and A. Majda. Absorbing boundary conditions for the numerical simulation of waves. *Math. Comp.*, 31:629–651, 1977.
 - [3] A. Bayliss and E. Turkel. Radiation boundary conditions for wave-like equations. *Comm. Pure and Appl. Math.*, 33:707–725, 1980.
 - [4] R.L. Higdon. Absorbing boundary conditions for difference approximations to the multi-dimensional wave equations. *Math. Comp.*, 47:437–459, 1986.
 - [5] D. Givoli. High-order nonreflecting boundary conditions without high-order derivatives. *J. Comp. Phys.*, 170:849–870, 2001.
 - [6] D. Givoli and B. Neta. High-order non-reflecting boundary scheme for time-dependent waves. *J. Comp. Phys.*, 186:24–46, 2003.
 - [7] B. Alpert, L. Greengard, and T. Hagstrom. Rapid evaluation of nonreflecting boundary kernels for time-domain wave propagation. *SIAM J. Numer. Anal.*, 37:1138–1164, 2000.
 - [8] J. Winicour. Characteristic evolution and matching. *Living Rev. Relativity*, 8, 2005. URL (cited on 28 Sep 2006): <http://www.livingreviews.org/lrr-2005-10>.
 - [9] M. Choptuik, L. Lehner, I. Olabarrieta, R. Petryk, F. Pretorius, and H. Villegas. Towards the final fate of an unstable black string. *Phys. Rev. D*, 68:044001(1)–044001(11), 2003.
 - [10] F. Pretorius. Evolution of binary black-hole spacetimes. *Phys. Rev. Lett.*, 95:121101(1)–121101(4), 2005.
 - [11] H. Friedrich. On the regular and asymptotic characteristic initial value problem for Einstein's vacuum field equations. *Proc. R. Soc. Lond. A*, 375:169–184, 1981.
 - [12] J. Frauendiener. Numerical treatment of the hyperboloidal initial value problem for the vacuum Einstein equations. 2. The evolution equations. *Phys. Rev. D*, 58:064003(1)–064003(18), 1998.
 - [13] S. Husa, C. Schneemann, T. Vogel, and A. Zenginoğlu. Hyperboloidal data and evolution. *AIP Conf. Proc.*, 841:306–313, 2006. 28th Spanish Relativity Meeting (ERE05): A Century of Relativity Physics, Oviedo, Asturias, Spain, 6–10 Sep 2005.
 - [14] W.L. Burke. Gravitational radiation damping of slowly moving systems calculated using matched asymptotic expansions. *J. Math. Phys.*, 12:401–418, 1971.
 - [15] L.T. Buchman and O.C.A. Sarbach. Towards absorbing outer boundaries in general relativity. *Class. Quantum Grav.*, 23:6709–6744, 2006.
 - [16] J. Novak and S. Bonazzola. Absorbing boundary conditions for simulation of gravitational waves with spectral methods in spherical coordinates. *J. Comp. Phys.*, 197:186–196, 2004.
 - [17] J.M. Bardeen and W.H. Press. Radiation fields in the Schwarzschild background. *J. Math. Phys.*, 14:7–19, 1973.
 - [18] L.T. Buchman and O.C.A. Sarbach. Improved outer boundary conditions for Einstein's field equations. *Class. Quantum Grav.*, 24:S307–S326, 2007.
 - [19] S.R. Lau. Rapid evaluation of radiation boundary kernels for time-domain wave propagation on blackholes: theory and numerical methods. *J. Comput. Phys.*, 199:376–422, 2004.
 - [20] S.R. Lau. Rapid evaluation of radiation boundary kernels for time-domain wave propagation on black holes: implementation and numerical tests. *Class. Quantum Grav.*, 21:4147–4192, 2004.
 - [21] S.R. Lau. Analytic structure of radiation boundary kernels for blackhole perturbations. *J. Math. Phys.*, 46:102503(1)–102503(21), 2005.
 - [22] J.M. Stewart. The Cauchy problem and the initial boundary value problem in numerical relativity. *Class. Quantum Grav.*, 15:2865–2889, 1998.
 - [23] H. Friedrich and G. Nagy. The initial boundary value problem for Einstein's vacuum field equations. *Comm. Math. Phys.*, 201:619–655, 1999.

- [24] B. Szilagyi, B.G. Schmidt, and J. Winicour. Boundary conditions in linearized harmonic gravity. *Phys. Rev. D*, 65:064015(1)–064015(15), 2002.
- [25] B. Szilagyi and J. Winicour. Well-posed initial-boundary evolution in general relativity. *Phys. Rev. D*, 68:041501(1)–041501(5), 2003.
- [26] G. Calabrese, J. Pullin, O. Reula, O. Sarbach, and M. Tiglio. Well posed constraint-preserving boundary conditions for the linearized Einstein equations. *Comm. Math. Phys.*, 240:377–395, 2003.
- [27] G. Calabrese and O. Sarbach. Detecting ill posed boundary conditions in general relativity. *J. Math. Phys.*, 44:3888–3899, 2003.
- [28] G. Calabrese. *Constraint preserving boundary conditions for the linearized Einstein equations*. PhD thesis, Louisiana State University, 2003. <http://etd.lsu.edu/docs/available/etd-1105103-100340>.
- [29] S. Frittelli and R. Gomez. Boundary conditions for hyperbolic formulations of the Einstein equations. *Class. Quantum Grav.*, 20:2379–2392, 2003.
- [30] S. Frittelli and R. Gomez. Einstein boundary conditions for the 3+1 Einstein equations. *Phys. Rev. D*, 68:044014(1)–044014(6), 2003.
- [31] S. Frittelli and R. Gomez. Einstein boundary conditions in relation to constraint propagation for the initial-boundary value problem of the Einstein equations. *Phys. Rev. D*, 69:124020(1)–124020(10), 2004.
- [32] S. Frittelli and R. Gomez. Einstein boundary conditions for the Einstein equations in the conformal-traceless decomposition. *Phys. Rev. D*, 70:064008(1)–064008(13), 2004.
- [33] C. Gundlach and J.M. Martín-García. Symmetric hyperbolic form of systems of second-order evolution equations subject to constraints. *Phys. Rev. D*, 70:044031(1)–044031(14), 2004.
- [34] C. Gundlach and J.M. Martín-García. Symmetric hyperbolic and consistent boundary conditions for second order Einstein equations. *Phys. Rev. D*, 70:044032(1)–044032(16), 2004.
- [35] O. Reula and O. Sarbach. A model problem for the initial-boundary value formulation of Einstein’s field equations. *Journal of Hyperbolic Differential Equations*, 2:397–435, 2005.
- [36] O. Sarbach and M. Tiglio. Boundary conditions for Einstein’s field equations: Mathematical and numerical analysis. *Journal of Hyperbolic Differential Equations*, 2:839–883, 2005.
- [37] L.E. Kidder, L. Lindblom, M.A. Scheel, L.T. Buchman, and H.P. Pfeiffer. Boundary conditions for the Einstein evolution system. *Phys. Rev. D*, 71:064020(1)–064020(22), 2005.
- [38] N. Tarfulea. *Constraint preserving boundary conditions for hyperbolic formulations of Einstein’s equations*. PhD thesis, Minnesota University, 2005. gr-qc/0508014.
- [39] L. Lindblom, M.A. Scheel, L.E. Kidder, R. Owen, and O. Rinne. A new generalized harmonic evolution system. *Class. Quantum Grav.*, 23:S447–S462, 2006.
- [40] G. Nagy and O. Sarbach. A minimization problem for the lapse and the initial-boundary value problem for Einstein’s field equations. *Class. Quantum Grav.*, 23:S477–S504, 2006.
- [41] H.O. Kreiss and J. Winicour. Problems which are well-posed in a generalized sense with applications to the Einstein equations. *Class. Quantum Grav.*, 23:S405–S420, 2006.
- [42] O. Rinne. Stable radiation-controlling boundary conditions for the generalized harmonic Einstein equations. *Class. Quantum Grav.*, 23:6275–6300, 2006.
- [43] D.N. Arnold and N. Tarfulea. Boundary conditions for the Einstein-christoffel formulation of Einstein’s equations, 2006. gr-qc/0611010.
- [44] A.M. Alekseenko. Well-posed initial-boundary value problem for a constrained evolution system and radiation-controlling constraint-preserving boundary conditions, 2006. gr-qc/0611011.
- [45] M.S. Iriondo and O.A. Reula. On free evolution of selfgravitating, spherically symmetric waves. *Phys. Rev. D*, 65:044024(1)–044024(11), 2002.
- [46] G. Calabrese, L. Lehner, and M. Tiglio. Constraint-preserving boundary conditions in numerical relativity. *Phys. Rev. D*, 65:104031(1)–104031(13), 2002.
- [47] J.M. Bardeen and L.T. Buchman. Numerical tests of evolution systems, gauge conditions, and boundary conditions for 1d colliding gravitational plane waves. *Phys. Rev. D*, 65:064037(1)–064037(23), 2002.
- [48] L. Lindblom, M.A. Scheel, H.P. Pfeiffer, L.E. Kidder, D. Shoemaker, and S.A. Teukolsky. Controlling the growth of constraints in hyperbolic evolution systems. *Phys. Rev. D*, 69:124025(1)–124025(14), 2004.
- [49] M. Holst, L. Lindblom, R. Owen, H.P. Pfeiffer, M.A. Scheel, and L.E. Kidder. Optimal constraint projection for hyperbolic evolution systems. *Phys. Rev. D*, 70:084017(1)–084017(17), 2004.
- [50] C. Bona, T. Ledvinka, C. Palenzuela-Luque, and M. Zacek. Constraint-preserving boundary conditions in the Z4 numerical relativity formalism. *Class. Quantum Grav.*, 22:2615–2634, 2005.
- [51] O. Rinne. *Axisymmetric numerical relativity*. PhD thesis, Cambridge University, 2005. gr-qc/0601064.
- [52] M.C. Babiuc, B. Szilagyi, and J. Winicour. Harmonic initial-boundary evolution in general relativity. *Phys. Rev. D*, 73:064017(1)–064017(23), 2006.
- [53] M.C. Babiuc, H.O. Kreiss, and J. Winicour. Constraint-preserving Sommerfeld conditions for the harmonic Einstein equations. *Phys. Rev. D*, 75:044002(1)–044002(13), 2007.
- [54] O. Rinne, L. Lindblom, and M.A. Scheel. Testing outer boundary treatments for the Einstein equations. *Class. Quantum Grav.*, 24:4053–4078, 2007.
- [55] M.A. Scheel, H.P. Pfeiffer, L. Lindblom, L.E. Kidder, O. Rinne, and S.A. Teukolsky. Solving Einstein’s equations with dual coordinate frames. *Phys. Rev. D*, 74:104006(1)–104006(13), 2006.
- [56] M. Ruiz, O. Rinne, and O. Sarbach. Outer boundary conditions for Einstein’s field equations in harmonic coordinates,

2007. arXiv:0707.2797 [gr-qc].
- [57] T. Regge and J. Wheeler. Stability of a Schwarzschild singularity. *Phys. Rev.*, 108:1063–1069, 1957.
 - [58] F. Zerilli. Effective potential for even-parity Regge-Wheeler gravitational perturbation equations. *Phys. Rev. Lett.*, 24:737–738, 1970.
 - [59] O. Sarbach and M. Tiglio. Gauge invariant perturbations of Schwarzschild black holes in horizon penetrating coordinates. *Phys. Rev. D*, 64:084016(1)–084016(15), 2001.
 - [60] J.M. Bardeen, 2007. Private communication.
 - [61] K.O. Friedrichs. Symmetric positive linear differential equations. *Commun. Pure Appl. Math.*, 11:333–418, 1958.
 - [62] P.D. Lax and R.S. Phillips. Local boundary conditions for dissipative symmetric linear differential operators. *Commun. Pure Appl. Math.*, 13:427–455, 1960.
 - [63] P. Secchi. Well-posedness of characteristic symmetric hyperbolic systems. *Arch. Rat. Mech. Anal.*, 134:155–197, 1996.
 - [64] H.O. Kreiss and J. Lorenz. *Initial-Boundary Value Problems and the Navier-Stokes Equations*. Academic Press, 1989.
 - [65] H.O. Kreiss. Initial boundary value problems for hyperbolic systems. *Commun. Pure Appl. Math.*, 23:277–298, 1970.
 - [66] H. O. Kreiss, O. Reula, O. Sarbach, and J. Winicour. The Einstein equations, boundaries and integration by parts, 2007. arXiv:0707.4188 [gr-qc].
 - [67] O. Sarbach and L. Lehner. No naked singularities in homogeneous, spherically symmetric bubble spacetimes? *Phys. Rev. D*, 69:021901(1)–021901(5), 2004.
 - [68] F.S. Guzmán, L. Lehner, and O. Sarbach. Do unbounded bubbles ultimately become fenced inside a black hole?, 2007. arXiv:0706.3915 [hep-th].
 - [69] M. Shibata and T. Nakamura. Evolution of three-dimensional gravitational waves: Harmonic slicing case. *Phys. Rev. D*, 52:5428–5444, 1995.
 - [70] T.W. Baumgarte and S.L. Shapiro. On the numerical integration of Einstein’s field equations. *Phys. Rev. D*, 59:024007(1)–024007(7), 1999.
 - [71] H. Beyer and O. Sarbach. On the well posedness of the Baumgarte-Shapiro-Shibata-Nakamura formulation of Einstein’s field equations. *Phys. Rev. D*, 70:104004(1)–104004(11), 2004.